COMMENT

On a potential with the spectrum of the hydrogen atom

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Abstract. Using an extension of a factorisation given by Ding using radial ladder operators, a one-parameter family of potentials with spectra coinciding with that of the hydrogen atom is generated.

Recently, Ding [1] solved the radial hydrogen atom by the factorisation method. He employed radial ladder operators for the factorisation. By this method he simplified the calculation of matrix elements and inner products. Here we extend the factorisation so that a potential with a spectrum the same as the hydrogen atom is obtained. The resulting potential may also be obtainable from the inverse scattering method, e.g., by the method of Abraham and Moses [2] using the Gelfand–Levitan procedure. The new factorisation is developed along the lines of Mielnik [3] or Zhu [4]. The general setting of 'shape-invariant' potentials is discussed in Dutt *et al* [5].

We take as the reduced radial Schrödinger equation with Coulomb potential

$$H_l U_l(r) \equiv \left(-\frac{d^2}{dr^2} - \frac{2}{r} + \frac{l(l+1)}{r^2} \right) U_l(r) = \varepsilon_l U_l(r) \tag{1}$$

subject to $U_l(0) = U_l(\infty) = 0$, $\int_0^\infty U_l^2(r)\,dr = 1$ where l is the angular momentum quantum number and we take the Bohr radius as the unit distance.

We define the two operators

$$a_l^\pm \equiv \pm\frac{d}{dr} + \frac{l+1}{r} - \frac{1}{l+1} - g_l(r) \tag{2}$$

with $g_l(r) \in C^1(R^+)$ to be determined such that the following factorisation still holds as in [1]:

$$H_l^+ \equiv a_l^+ a_l^- = H_l + \frac{1}{(l+1)^2}. \tag{3}$$

We find that $g_l(r)$ must satisfy the differential equation

$$\frac{dg_l}{dr} + 2\left(\frac{l+1}{r} - \frac{1}{l+1} \right) g_l - g_l^2 = 0. \tag{4}$$

This Ricatti equation can be solved with the substitution $g_l = 1/y$, with the result

$$g_l(r) = -r^{-2(l+1)} \exp\left(\frac{2}{(l+1)} r \right) \left[\int_\delta^r t^{-2(l+1)} \exp\left(\frac{2}{(l+1)} t \right) dt + c \right]^{-1} \tag{5}$$

where c is a constant and $\delta > 0$. This gives $a_l^{\pm}$ in (2), which continue to be adjoint to one another. Obviously, as the parameter $c \to \infty$ in (5) the factorisation in [1] is recovered.

We now find that

$$H_l^- \equiv a_l^- a_l^+ = H_{l+1} + \frac{1}{(l+1)^2} + 2\frac{dg_l}{dr}$$

$$= H_l + \frac{2(l+1)}{r^2} + \frac{1}{(l+1)^2} + 2\frac{dg_l}{dr} \tag{6}$$

so that the commutator of a_l^+ and a_l^- is given by

$$[a_l^+, a_l^-] = -2\frac{dg_l}{dr} - \frac{2(l+1)}{r^2}. \tag{7}$$

In fact we have

$$[a_l^+, a_{l'}^-] = -\frac{d}{dr}(g_l + g_{l'}) - \frac{(l' + l + 2)}{r^2} \tag{8a}$$

$$[a_l^+, a_{l'}^+] = \frac{d}{dr}(g_l - g_{l'}) + \frac{(l - l')}{r^2} \tag{8b}$$

$$[a_l^-, a_{l'}^-] = \frac{d}{dr}(g_{l'} - g_l) + \frac{(l' - l)}{r^2} \tag{8c}$$

which extends results in [1]. These relations allow us to calculate

$$[H_l, a_l^+] = 2a_l^+ \left(\frac{dg_l}{dr} + \frac{l+1}{r^2}\right) \tag{9a}$$

$$[H_l, a_l^-] = -2\left(\frac{dg_l}{dr} + \frac{l+1}{r^2}\right)a_l^-. \tag{9b}$$

Using the factorisations in (3) and (6) we can construct a one-parameter family of potentials with spectra coinciding with that of the hydrogen atom. We set

$$\tilde{H}_l \equiv a_l^- a_l^+ - \frac{1}{(l+1)^2} = H_l + V'(r) \tag{10}$$

where we define

$$V'(r) \equiv \frac{2(l+1)}{r^2} + 2\frac{dg_l}{dr}. \tag{11}$$

Then, using (3),

$$\tilde{H}_l a_l^- = \left(a_l^- a_l^+ - \frac{1}{(l+1)^2}\right)a_l^- = a_l^- \left(a_l^- a_l^+ - \frac{1}{(l+1)^2}\right)$$

$$= a_l^- H_l. \tag{12}$$

So, by (1), $\tilde{H}_l(a_l^- U_l) = \varepsilon_l(a_l^- U_l)$. Therefore $\tilde{H}_l$ and H_l have the same spectra and the eigenvectors of $\tilde{H}_l$ are

$$\tilde{U}_l = \tilde{N}a_l^- U_l \tag{13}$$

where $\tilde{N}$ is a constant of normalisation. H_l and $\tilde{H}_l$ have the same energies aside from the ground state of H_l. The ground state of H_l is determined by $a_l^- U_l^{(0)} = 0$ and that of $\tilde{H}_l$ by $a_l^+ \tilde{U}_l^{(0)} = 0$. For example, the solution of $a_l^- U_l^{(0)} = 0$ is

$$U_l^{(0)}(r) = N \exp\left(-\frac{1}{(l+1)}r\right) r^{l+1} \exp\left(-\int^r g_l(t)\,\mathrm{d}t\right) \tag{14}$$

which is normalisable.

The normalisation of the functions $\{\tilde{U}_l\}$ can be obtained with the aid of the adjointness of a_l^+ and a_l^-:

$$(1/\tilde{N}^2)\langle \tilde{U}_l, \tilde{U}_l\rangle = \langle a_l^- U_l, a_l^- U_l\rangle = \langle U_l, a_l^+ a_l^- U_l\rangle$$

$$= \left\langle U_l, \left(H_l + \frac{1}{(l+1)^2}\right) U_l\right\rangle = \left(\varepsilon_l + \frac{1}{(l+1)^2}\right)\langle U_l, U_l\rangle \tag{15}$$

so that $\tilde{N} = (l+1)[1+(l+1)^2\varepsilon_l]^{-1/2}$.

Further implications of the above relations, comparison with the inverse scattering method, including the method of Abraham and Moses [2], and application to supersymmetric quantum mechanics will be discussed elsewhere.

References

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